Number Theory 17

09 March 2024 11:13

Q> Let m and n be positive integers, such that,
$\gcd(11k-1, m) = \gcd(11k-1, n)$ holds for all positive integer k.
Prove that $m = 11^r n$ for some integer r.

Ans:— Let $p \neq 11$ be a prime such that $p^a \mid m$ but $p^a \nmid n$

Let $m = p^a b$, $n = p^c d$ where b and d are coprime to p, $a > c$

$\exists$ a k such that $11k \equiv 1 \pmod{p^a}$ and $p^a \mid \gcd(11k-1, m)$

$$\Rightarrow p^a \mid \gcd(11k-1, n) \Rightarrow \Leftarrow$$ Contradiction

$\boxed{\text{Similarly for } a < c \text{ case}}$ → not needed as r is any integer.

Hence $a = c$ $\forall p \neq 11$. Thus $m = 11^r n$.

Q> How many primes p are there such that $29^p + 1$ is a multiple of p?

Ans:— If $p = 29k$ then $\gcd(29, p) \neq 1$ and $29 \nmid 29^p + 1$

$\Rightarrow 29k \nmid 29^p + 1$

$\Rightarrow p \nmid 29^p + 1$

Now if $p \neq 29k$, then

$\gcd(p, 29) = 1$ as 29 is prime.

$30 = 2 \times 3 \times 5$

Then $29^p \equiv 29 \pmod p$ [Fermat's Little Theorem]

$\Rightarrow 29^p + 1 \equiv 30 \pmod p \equiv 0 \pmod p$

$\Rightarrow p \mid 30 \Rightarrow p \in \{2, 3, 5\}$

Q> Calculate the last three digits of $2005^{11} + 2005^{12} + \cdots + 2005^{2006}$.

Ans:— $2005 \equiv 5 \pmod{1000}$

$5^{11} + 5^{12} + \cdots + 5^{2006} \pmod{1000}$

$5^{11} + 5^{12} + \cdots + 5^{2006} \equiv 0 \pmod{125}$

$5^{2k+1} \equiv 5 \pmod 8$

$$5^{11} + 5^{12} + \cdots + 5^{2006} \equiv 0 \pmod{125}$$

$$5^{2k} \equiv 1 \pmod 8 \qquad 5^{2k+1} \equiv 5 \pmod 8$$

$$\Rightarrow 5^{11} + 5^{12} + \cdots + 5^{2006}$$

$$\equiv 998(1+5) \equiv 998 \times 6 \pmod 8$$

$$\equiv 4 \pmod 8$$

$$125 \equiv 5 \pmod 8$$

$$\Rightarrow 125 \times 4 \equiv 20 \pmod 8$$

$$\Rightarrow 500 \equiv 4 \pmod 8$$

$$\Rightarrow 5^{11} + \cdots + 5^{2006} \equiv 500 \pmod{1000}$$

HomeWork

1) Let a and b be relatively prime positive integers. Prove that there are infinitely many relatively prime terms in the AP, $a, a+b, a+2b, a+3b, \ldots$

2) Evaluate $\left\lfloor \frac{2^0}{3} \right\rfloor + \left\lfloor \frac{2^1}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \cdots + \left\lfloor \frac{2^{1000}}{3} \right\rfloor$

3) Find the last two digits of 1032^{1032}.